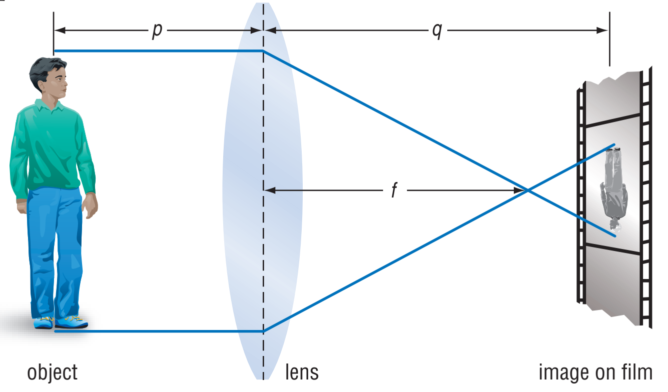
8-2 Adding and Subtracting Rational Expressions

Main Ideas

- Determine the LCM of polynomials.
- Add and subtract rational expressions.

GET READY for the Lesson

In order to produce a picture that is "in focus," the distance between the camera lens and the film q must be controlled so that it satisfies a particular relationship. If the distance from the subject to the lens is p and the focal length of the lens is f, then the formula $\frac{1}{q} = \frac{1}{f} - \frac{1}{p}$ can be used to determine the correct distance between the lens and the film.



LCM of Polynomials To find $\frac{5}{6} - \frac{1}{4}$ or $\frac{1}{f} - \frac{1}{p}$, you must first find the least common denominator (LCD). The LCD is the least common multiple (LCM) of the denominators.

To find the LCM of two or more numbers or polynomials, factor each number or polynomial. The LCM contains each factor the greatest number of times it appears as a factor.

LCM of 6 and 4

$6 = 2 \cdot 3$
$4 = 2^2$
$\text{LCM} = 2^2 \cdot 3 \text{ or } 12$

LCM of $a^2 - 6a + 9$ and $a^2 + a - 12$

$a^2 - 6a + 9 = (a - 3)^2$
$a^2 + a - 12 = (a - 3)(a + 4)$
$\text{LCM} = (a - 3)^2(a + 4)$

EXAMPLE LCM of Monomials

1 **Find the LCM of $18r^2s^5$, $24r^3st^2$, and $15s^3t$.**

$18r^2s^5 = 2 \cdot 3^2 \cdot r^2 \cdot s^5$ — Factor the first monomial.

$24r^3st^2 = 2^3 \cdot 3 \cdot r^3 \cdot s \cdot t^2$ — Factor the second monomial.

$15s^3t = 3 \cdot 5 \cdot s^3 \cdot t$ — Factor the third monomial.

$\text{LCM} = 2^3 \cdot 3^2 \cdot 5 \cdot r^3 \cdot s^5 \cdot t^2$ — Use each factor the greatest number of times it appears as a factor and simplify.
$= 360r^3s^5t^2$

CHECK Your Progress

Find the LCM of each set of monomials.

1A. $12a^2b^4$, $27ac^3$, $18a^5b^2c$

1B. $6m^3n^5$, $42mnp^2$, $36m^3n^4p$

EXAMPLE 2 LCM of Polynomials

Find the LCM of $p^3 + 5p^2 + 6p$ and $p^2 + 6p + 9$.

$p^3 + 5p^2 + 6p = p(p + 2)(p + 3)$ Factor the first polynomial.

$p^2 + 6p + 9 = (p + 3)^2$ Factor the second polynomial.

$\text{LCM} = p(p + 2)(p + 3)^2$ Use each factor the greatest number of times it appears as a factor.

CHECK Your Progress

Find the LCM of each set of polynomials.

2A. $q^2 - 4q + 4$ and $q^3 - 3q^2 + 2q$

2B. $2k^3 - 5k^2 - 12k$ and $k^3 - 8k^2 + 16k$

Add and Subtract Rational Expressions

As with fractions, to add or subtract rational expressions, you must have common denominators.

Specific Case		General Case
$\frac{2}{3} + \frac{3}{5} = \frac{2 \cdot 5}{3 \cdot 5} + \frac{3 \cdot 3}{5 \cdot 3}$	Find equivalent fractions that have a common denominator.	$\frac{a}{c} + \frac{b}{d} = \frac{a \cdot d}{c \cdot d} + \frac{b \cdot c}{d \cdot c}$
$= \frac{10}{15} + \frac{9}{15}$	Simplify each numerator and denominator.	$= \frac{ad}{cd} + \frac{bc}{cd}$
$= \frac{19}{15}$	Add the numerators.	$= \frac{ad + bc}{cd}$

As with fractions, you can use the least common multiple of the denominators to find the least common denominator for two rational expressions.

EXAMPLE 3 Monomial Denominators

Simplify $\frac{7x}{15y^2} + \frac{y}{18xy}$.

$\frac{7x}{15y^2} + \frac{y}{18xy} = \frac{7x \cdot 6x}{15y^2 \cdot 6x} + \frac{y \cdot 5y}{18xy \cdot 5y}$ The LCD is $90xy^2$. Find the equivalent fractions that have this denominator.

$= \frac{42x^2}{90xy^2} + \frac{5y^2}{90xy^2}$ Simplify each numerator and denominator.

$= \frac{42x^2 + 5y^2}{90xy^2}$ Add the numerators.

CHECK Your Progress

Simplify each expression.

3A. $\frac{8a}{9b} - \frac{1}{7ab^2}$

3B. $\frac{1}{8m^2n} + \frac{2}{mn^2}$

3C. $\frac{2}{3xy} - \frac{3x}{5y}$

3D. $\frac{6c}{7b^2} + \frac{2d}{3ab}$

EXAMPLE 4 Polynomial Denominators

Study Tip

Common Factors

Sometimes when you simplify the numerator, the polynomial contains a factor common to the denominator. Thus the rational expression can be further simplified.

4 Simplify $\frac{w+12}{4w-16} - \frac{w+4}{2w-8}$.

$$\frac{w+12}{4w-16} - \frac{w+4}{2w-8} = \frac{w+12}{4(w-4)} - \frac{w+4}{2(w-4)}$$ Factor the denominators.

$$= \frac{w+12}{4(w-4)} - \frac{(w+4)(2)}{2(w-4)(2)}$$ The LCD is $4(w-4)$.

$$= \frac{(w+12)-(2)(w+4)}{4(w-4)}$$ Subtract the numerators.

$$= \frac{w+12-2w-8}{4(w-4)}$$ Distributive Property

$$= \frac{-w+4}{4(w-4)}$$ Combine like terms.

$$= \frac{-1\cancel{(w-4)}^{1}}{4\cancel{(w-4)}_{1}} \text{ or } -\frac{1}{4}$$ Simplify.

CHECK Your Progress Simplify each expression.

4A. $\frac{x+6}{6x-18} + \frac{x-6}{2x-6}$

4B. $\frac{x-1}{3x^2+8x+5} - \frac{x-1}{12x+20}$

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One way to simplify a complex fraction is to simplify the numerator and the denominator separately, and then simplify the resulting expressions.

EXAMPLE 5 Simplify Complex Fractions

5 Simplify $\frac{\frac{1}{x} - \frac{1}{y}}{1 + \frac{1}{x}}$.

$$\frac{\frac{1}{x} - \frac{1}{y}}{1 + \frac{1}{x}} = \frac{\frac{y}{xy} - \frac{x}{xy}}{\frac{x}{x} + \frac{1}{x}}$$ The LCD of the numerator is xy. The LCD of the denominator is x.

$$= \frac{\frac{y-x}{xy}}{\frac{x+1}{x}}$$ Simplify the numerator and denominator.

$$= \frac{y-x}{xy} \div \frac{x+1}{x}$$ Write as a division expression.

$$= \frac{y-x}{\cancel{x}y} \cdot \frac{\cancel{x}}{x+1}$$ Multiply by the reciprocal of the divisor.

$$= \frac{y-x}{y(x+1)} \text{ or } \frac{y-x}{xy+y}$$ Simplify.

CHECK Your Progress Simplify each expression.

5A. $\frac{\frac{1}{y} + \frac{1}{x}}{\frac{1}{y} - \frac{1}{x}}$

5B. $\frac{\frac{a}{b} + 1}{1 - \frac{b}{a}}$

EXAMPLE 6 Use a Complex Fraction to Solve a Problem

COORDINATE GEOMETRY Find the slope of the line that passes through $A\left(\frac{2}{p}, \frac{1}{2}\right)$ and $B\left(\frac{1}{3}, \frac{3}{p}\right)$.

$m = \frac{y_2 - y_1}{x_2 - x_1}$ — Definition of slope

$= \frac{\frac{3}{p} - \frac{1}{2}}{\frac{1}{3} - \frac{2}{p}}$ — $y_2 = \frac{3}{p}$, $y_1 = \frac{1}{2}$, $x_2 = \frac{1}{3}$, and $x_1 = \frac{2}{p}$

$= \frac{\frac{6-p}{2p}}{\frac{p-6}{3p}}$ — The LCD of the numerator is $2p$. The LCD of the denominator is $3p$.

$= \frac{6-p}{2p} \div \frac{p-6}{3p}$ — Write as a division expression.

$= \frac{\overset{-1}{\cancel{6-p}}}{\underset{1}{2\cancel{p}}} \cdot \frac{\overset{1}{3\cancel{p}}}{\underset{1}{\cancel{p-6}}}$ or $-\frac{3}{2}$ The slope is $-\frac{3}{2}$.

Study Tip

Check Your Solution

You can check your answer by letting p equal any nonzero number, say 1. Use the definition of slope to find the slope of the line through the points.

CHECK Your Progress

Find the slope of the line that passes through each pair of points.

6A. $C\left(\frac{1}{4}, \frac{4}{q}\right)$ and $D\left(\frac{5}{q}, \frac{1}{5}\right)$

6B. $E\left(\frac{7}{w}, \frac{1}{7}\right)$ and $F\left(\frac{1}{7}, \frac{7}{w}\right)$

CHECK Your Understanding

Examples 1, 2 (pp. 450–451)

Find the LCM of each set of polynomials.

1. $12y^2, 6x^2$
2. $16ab^3, 5b^2a^2, 20ac$
3. $x^2 - 2x, x^2 - 4$
4. $x^3 - 4x^2 - 5x, x^2 + 6x + 5$

Simplify each expression.

Example 3 (p. 451)

5. $\frac{2}{x^2y} - \frac{x}{y}$
6. $\frac{7a}{15b^2} - \frac{b}{18ab}$
7. $\frac{5}{3m} - \frac{2}{7m} - \frac{1}{2m}$
8. $\frac{3x}{5} - \frac{1}{2x^2} + \frac{3}{4x}$

Example 4 (p. 452)

9. $\frac{6}{d^2 + 4d + 4} + \frac{5}{d + 2}$
10. $\frac{a}{a^2 - a - 20} + \frac{2}{a + 4}$
11. $\frac{1}{x^2 - 4} + \frac{x}{x + 2}$
12. $\frac{x}{x + 1} + \frac{3}{x^2 - 4x - 5}$

Example 5 (p. 452)

13. $\frac{x + \frac{x}{3}}{x - \frac{x}{6}}$
14. $\frac{1 - \frac{1}{x}}{x - \frac{1}{x}}$
15. $\frac{2 - \frac{4}{x}}{x - \frac{4}{x}}$
16. $\frac{x - \frac{x}{2}}{x + \frac{x}{8}}$

Example 6 (p. 453)

17. **GEOMETRY** An expression for the area of a rectangle is $4x + 16$. Find the width of the rectangle. Express in simplest form.

$\frac{x + 4}{x + 2}$

Exercises

HOMEWORK HELP

For Exercises	See Examples
18, 19	1
20, 21	2
22–25	3
26–31	4
32, 33	5
34, 35	6

Find the LCM of each set of polynomials.

18. $10s^2, 35s^2t^2$

19. $36x^2y, 20xyz$

20. $4w - 12, 2w - 6$

21. $x^2 - y^2, x^3 + x^2y$

Simplify each expression.

22. $\frac{6}{ab} + \frac{8}{a}$

23. $\frac{5}{6v} + \frac{7}{4v}$

24. $\frac{3x}{4y^2} - \frac{y}{6x}$

25. $\frac{5}{a^2b} - \frac{7a}{5a^2}$

26. $\frac{7}{y-8} - \frac{6}{8-y}$

27. $\frac{a}{a-4} - \frac{3}{4-a}$

28. $\frac{m}{m^2-4} + \frac{2}{3m+6}$

29. $\frac{y}{y+3} - \frac{6y}{y^2-9}$

30. $\frac{5}{x^2-3x-28} + \frac{7}{2x-14}$

31. $\frac{d-4}{d^2+2d-8} - \frac{d+2}{d^2-16}$

32. $\dfrac{\frac{1}{b+2} + \frac{1}{b-5}}{\frac{2b^2-b-3}{b^2-3b-10}}$

33. $\dfrac{(x+y)\left(\frac{1}{x} - \frac{1}{y}\right)}{(x-y)\left(\frac{1}{x} + \frac{1}{y}\right)}$

34. GEOMETRY An expression for the length of one rectangle is $\frac{x^2-9}{x-2}$. The length of a similar rectangle is expressed as $\frac{x+3}{x^2-4}$. What is the scale factor of the two rectangles? Write in simplest form.

35. GEOMETRY Find the slope of a line that contains the points $A\left(\frac{1}{p}, \frac{1}{q}\right)$ and $B\left(\frac{1}{q}, \frac{1}{p}\right)$. Write in simplest form.

Find the LCM of each set of polynomials.

36. $14a^3, 15bc^3, 12b^3$

37. $9p^2q^3, 6pq^4, 4p^3$

38. $2t^2 + t - 3, 2t^2 + 5t + 3$

39. $n^2 - 7n + 12, n^2 - 2n - 8$

Simplify each expression.

40. $\frac{5}{r} + 7$

41. $\frac{2x}{3y} + 5$

42. $\frac{3}{4q} - \frac{2}{5q} - \frac{1}{2q}$

43. $\frac{11}{9} - \frac{7}{2w} - \frac{6}{5w}$

44. $\frac{1}{h^2-9h+20} - \frac{5}{h^2-10h+25}$

45. $\frac{x}{x^2+5x+6} - \frac{2}{x^2+4x+4}$

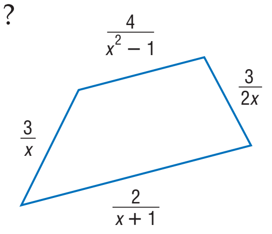
46. $\frac{m^2+n^2}{m^2-n^2} + \frac{m}{n-m} + \frac{n}{m+n}$

47. $\frac{y+1}{y-1} + \frac{y+2}{y-2} + \frac{y}{y^2-3y+2}$

48. Write $\left(\frac{2s}{2s+1} - 1\right) \div \left(1 + \frac{2s}{1-2s}\right)$ in simplest form.

49. What is the simplest form of $\left(3 + \frac{5}{a+2}\right) \div \left(3 - \frac{10}{a+7}\right)$?

50. GEOMETRY Find the perimeter of the quadrilateral. Express in simplest form.

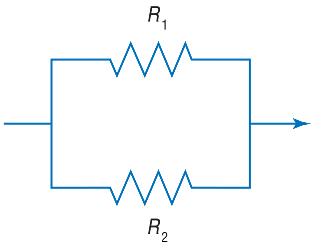


Real-World Link

The Tour de France is the most popular bicycle road race. It lasts 21 days and covers 2500 miles.

Source: *World Book Encyclopedia*

ELECTRICITY **For Exercises 51 and 52, use the following information.**
In an electrical circuit, if two resistors with resistance R_1 and R_2 are connected in parallel as shown, the relationship between these resistances and the resulting combination resistance R is $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.



51. If R_1 is x ohms and R_2 is 4 ohms less than twice x ohms, write an expression for $\frac{1}{R}$.

52. A circuit with two resistors connected in parallel has an effective resistance of 25 ohms. One of the resistors has a resistance of 30 ohms. Find the resistance of the other resistor.

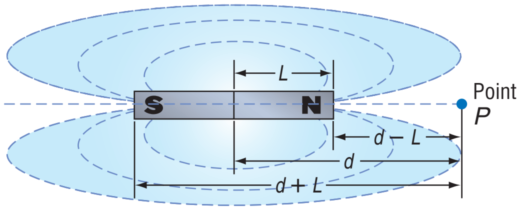
BICYCLING **For Exercises 53–55, use the following information.**
Jalisa is competing in a 48-mile bicycle race. She travels half the distance at one rate. The rest of the distance, she travels 4 miles per hour slower.

53. If x represents the faster pace in miles per hour, write an expression that represents the time spent at that pace.

54. Write an expression for the time spent at the slower pace.

55. Write an expression for the time Jalisa needed to complete the race.

56. **MAGNETS** For a bar magnet, the magnetic field strength H at a point P along the axis of the magnet is $H = \frac{m}{2L(d-L)^2} - \frac{m}{2L(d+L)^2}$. Write a simpler expression for H.



EXTRA PRACTICE
See pages 908, 933.

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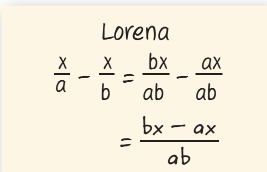
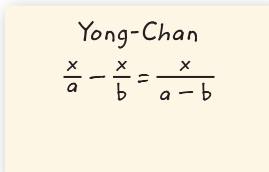
H.O.T. Problems

57. **OPEN ENDED** Write two polynomials that have a LCM of $d^3 - d$.

58. **FIND THE ERROR** Lorena and Yong-Chan are simplifying $\frac{x}{a} - \frac{x}{b}$. Who is correct? Explain your reasoning.





59. **CHALLENGE** Find two rational expressions whose sum is $\frac{2x - 1}{(x + 1)(x - 2)}$.

60. **REASONING** In the expression $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, a, b, and c are nonzero real numbers. Determine whether each statement is *sometimes*, *always*, or *never* true. Explain your answer.

a. abc is a common denominator.

b. abc is the LCD.

c. ab is the LCD.

d. b is the LCD.

e. The sum is $\frac{bc + ac + ab}{abc}$.

61. *Writing in Math* Use the information on page 450 to explain how subtraction of rational expressions is used in photography. Include an equation that could be used to find the distance between the lens and the film if the focal length of the lens is 50 millimeters and the distance between the lens and the object is 1000 millimeters.

STANDARDIZED TEST PRACTICE

62. **ACT/SAT** What is the sum of $\frac{x-y}{5}$ and $\frac{x+y}{4}$?

A $\frac{x+9y}{20}$

B $\frac{9x+y}{20}$

C $\frac{9x-y}{20}$

D $\frac{x-9y}{20}$

63. **REVIEW**

Given: Two angles are complementary. The measure of one angle is 15° more than the measure of the other angle.
Conclusion: The measures of the angles are 30° and 45°.

This conclusion —

F is contradicted by the first statement given.

G is verified by the first statement given.

H invalidates itself because a 45° angle cannot be complementary to another.

J verifies itself because 30° is 15° less than 45°.

Spiral Review

Simplify each expression. (Lesson 8-1)

64. $\frac{9x^2y^3}{(5xyz)^2} \div \frac{(3xy)^3}{20x^2y}$

65. $\frac{5a^2-20}{2a+2} \cdot \frac{4a}{10a-20}$

66. Graph $y \leq \sqrt{x+1}$. (Lesson 7-7)

Find all of the zeros of each function. (Lesson 6-9)

67. $g(x) = x^4 - 8x^2 - 9$

68. $h(x) = 3x^3 - 5x^2 + 13x - 5$

69. **GARDENS** Helene Jonson has a rectangular garden 25 feet by 50 feet. She wants to increase the garden on all sides by an equal amount. If the area of the garden is to be increased by 400 square feet, by how much should each dimension be increased? (Lesson 5-5)

70. Three times a number added to four times a second number is 22. The second number is two more than the first number. Find the numbers. (Lesson 3-2)

GET READY for the Next Lesson

PREREQUISITE SKILL **Factor each polynomial.** (Lesson 5-3)

71. $x^2 + 3x + 2$

72. $x^2 - 6x + 5$

73. $x^2 + 11x - 12$

74. $x^2 - 16$

75. $3x^2 - 75$

76. $x^3 - 3x^2 + 4x - 12$